Integration, Fundamental Theorem of Calculus

December 2, 2016

Problems

Problem 1. Find the following.

1. $\int_0^{2\pi} \cos(x)dx$

   Solution: $\int_0^{2\pi} \cos(x)dx = \sin(x) \mid_0^{2\pi} = 0$

2. The **unsigned** area bounded by $\cos(x)$ between 0 and 2π.

   Solution: Since $\cos x \leq 0$ for $x \in [\pi/2, 3\pi/2]$, the unsigned area equals to

$$\int_0^{\pi/2} \cos(x)dx + \int_{\pi/2}^{3\pi/2} -\cos(x)dx + \int_{3\pi/2}^{2\pi} \cos(x)dx = \sin(\pi/2) - \sin(0) - (\sin(3\pi/2) - \sin(\pi/2)) + \sin(2\pi) - \sin(3\pi/2) = 4.$$

3. $\int \frac{1}{x^2} \sin\left(\frac{1}{x}\right) dx$

   Solution: Make substitution $u = \frac{1}{x}$. Then $du = -\frac{1}{x^2}dx$, and so

$$\int \frac{1}{x^2} \sin\left(\frac{1}{x}\right) dx = \int -\sin(u)du = \cos u + C = \cos\left(\frac{1}{x}\right) + C$$

4. $\int_{-1}^{1} t^3(1+t^4)^3 dt$

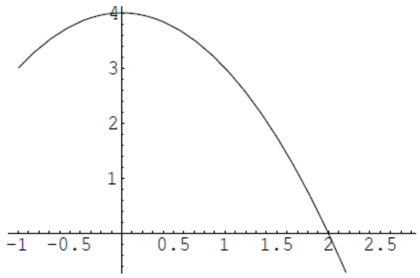
   Solution: since $t^3(1+t^4)^3$ is an odd function, and the interval $[-1, 1]$ is symmetric with respect to $t = 0$, the integral equals to 0. A different approach would be to use the substitution $u = t^4$.

5. $\int_0^{\pi/4} \tan x dx$

   Solution: Let $u = \cos(x)$. Then $du = -\sin(x)dx$, u changes from $\cos(0) = 1$ to $\cos(\pi/4) = 1/\sqrt{2}$, and

$$\int_0^{\pi/4} \tan x dx = -\int_1^{1/\sqrt{2}} \frac{du}{u} = \int_{1/\sqrt{2}}^{1} \frac{du}{u} = \ln(u) \mid_{1/\sqrt{2}}^{1} = \ln\sqrt{2}$$

Problem 2. Below is the graph of a function f.



Let $g(x) = \int_0^x f(t)dt$. Find $g(0), g'(0)$ and $g'(2)$.

Solution: $g(0) = \int_0^0 f(t)dt = 0$. By the Fundamental theorem of Calculus, $g'(0) = f(0) = 4$ and $g'(2) = f(2) = 0$.

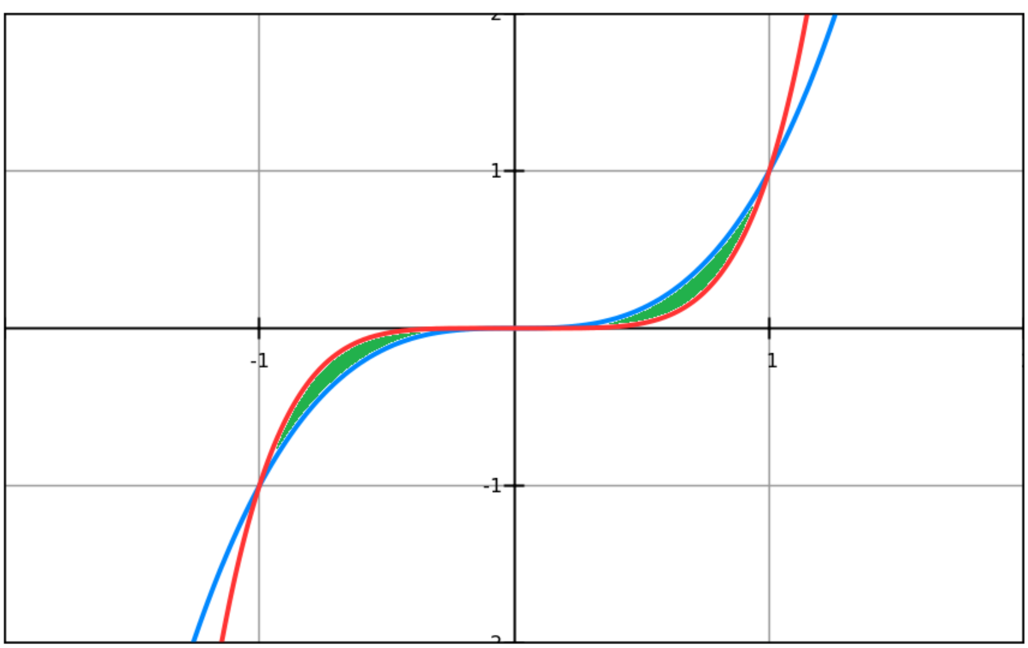
For $0 < x < 2$ the function $g(x)$ is

1. increasing and concave up;
2. increasing and concave down;
3. decreasing and concave up;
4. decreasing and concave down.

Solution: Since $g'(x) = f(x)$ is positive on $(0, 2)$, the function $g(x)$ is increasing. Since $g'(x) = f(x)$ is decreasing on $(0, 2)$, the function $g(x)$ is concave down.

Problem 3. Find the area of the propeller-shaped region enclosed by the curves $x - y^{1/3} = 0$ and $x - y^{1/5} = 0$.

Solution: The two graphs intersect at $(-1, -1)$, $(0, 0)$ and $(1, 1)$. The sketch is as follows (the area of interest is green):



Thus, the area of the propeller is given by

$$\int_{-1}^{0} y^{1/3} - y^{1/5} dy + \int_0^1 y^{1/5} - y^{1/3} dy = \frac{3}{4}(0-1) - \frac{5}{6}(0-1) + \frac{5}{6}(1-0) - \frac{3}{4}(1-0) = \frac{5}{3} - \frac{3}{2} = \frac{1}{6}$$

Problem 4. Let $f(x) = \int_{x^2}^{x^3} (t^2 - t)^2 dt$. Find $f'(x)$.

Solution: We can re-write $f(x)$ as $f(x) = \int_0^{x^3} (t^2 - t)^2 dt - \int_0^{x^2} (t^2 - t)^2 dt$. Then by the Fundamental Theorem of Calculus combined with the chain rule we get

$$f'(x) = (x^6 - x^3)^2 \cdot 3x^2 - (x^4 - x^2)^2 \cdot 2x$$

Problem 5. A rocket lifts o the surface of Earth with a constant acceleration of 20 m/sec^2. How fast will the rocket be going 1 minute later?

Solution: Acceleration is given by the derivative of the velocity, $a(t) = \frac{dv(t)}{dt}$. We are given $a(t) = \frac{dv(t)}{dt} = 20$ and so $v(t) = 20t + C$ for some constant C. Since at the time $t = 0$ the rocket is not moving, $v(0) = 0$, i.e. $C = 0$. This gives $v(t) = 20t$. In one minute, the speed will be $v(60) = 1200$ m/sec^2.

Problem 6. Compute the integral $\int \sqrt{1 - x^2} dx$. (Hint: $u = \arcsin x$ means $x = \sin u$.)

Use it to compute $\int_{-1}^{1} \sqrt{1 - x^2} dx$. Does the result match what you would expect from the usual geometric considerations?

Solution: Making the substitution $u = \arcsin x$, or $x = \sin u$, we get $\sqrt{1 - x^2} = \cos u$ and $dx = \cos u \, du$. Thus,

$$\int \sqrt{1 - x^2} dx = \int \cos^2 u \, du = \int \frac{1 + \cos 2u}{2} du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C = \frac{1}{2}\arcsin x + \frac{1}{4}\sin(2\arcsin x) + C$$

One last step is to simplify $\sin(2\arcsin x)$. We have

$$\sin(2\arcsin x) = 2\sin(\arcsin x)\cos(\arcsin x) = 2x\sqrt{1 - x^2}$$

Using the Fundamental Theorem of Calculus, $\int_{-1}^{1} \sqrt{1 - x^2} dx = \frac{\pi}{2}$. Since the graph of $\sqrt{1 - x^2}$ over $[-1, 1]$ is just a semi-circle of radius 1, its area is $\frac{\pi}{2}$ which is what we've got.

Problem 7. Using definite integrals, find the limit of the following sum:

$$\lim_{n \to \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \right)$$

(Hint: $\frac{1}{n+i} = \frac{1}{n} \cdot \frac{1}{1 + \frac{i}{n}}$)

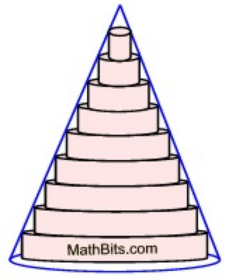
Solution: We need to understand $\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n+i} = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n} \cdot \frac{1}{1+\frac{i}{n}}$.

Consider the function $f(x) = \frac{1}{1+x}$ on the interval $[0, 1]$. It is continuous on this interval, and therefore integrable. The sum we are interested in, $\sum_{i=1}^{n} \frac{1}{n} \cdot \frac{1}{1+\frac{i}{n}}$ is just the lower sum for $f(x)$ on $[0, 1]$. Since f is integrable,

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n} \cdot \frac{1}{1 + \frac{i}{n}} = \int_0^1 \frac{dx}{x+1} = \ln(1 + x) \,|_0^1 = \ln(2)$$

So the answer is $\ln(2)$.

Problem 8. Using **Riemann sums**, find the formula for computing the volume of a cone of height h and radius r. You can use the formula for the volume of a cylinder.



Solution: Split $[0, h]$ into n intervals of equal length $\frac{h}{n}$. The radius r_i of the i-th thin cylinder can be found using similar triangles: $\frac{r_i}{r} = \frac{h - ih/n}{h} = 1 - i/n$, and so $r_i = r(1 - i/n)$. Therefore, the volume of the i-th thin cylinder is equal to $\frac{h}{n} \cdot \pi r^2 \left(1 - \frac{i}{n}\right)^2$ and the overall volume of all the thin cylinders is given by $\sum_{i=1}^{n} \frac{h}{n} \cdot \pi r^2 \left(1 - \frac{i}{n}\right)^2$. We need to compute the limit of this sum as $n \to \infty$.
This sum is a Riemann sum for the function $f(x) = \pi h r^2 (1 - x)^2$ on the interval $[0, 1]$. Therefore, the desired limit is equal to

$$\int_0^1 \pi h r^2 (1 - x)^2 dx = \pi h r^2 \cdot \int_0^1 (1 - x)^2 dx = \frac{1}{3} \pi h r^2$$

This is the same as the usual formula we know and love.